ON A MONOTONE SINGULAR FUNCTION AND ON THE APPROXIMATION OF ANALYTIC FUNCTIONS BY NEARLY ANALYTIC FUNCTIONS IN THE COMPLEX DOMAIN

BY

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The classical example of a monotone continuous function $y=\omega(t)$ which is singular(1) was generalized by T. Carleman(2). It was discussed in detail by E. Hille and J. D. Tamarkin(3), it was generalized further by R. E. Gilman(4) who dealt with the intricate properties of $\omega'(t)$; the method of these writers was based on binary, ternary, or higher scales of notation. In the present paper the Carleman-Gilman function will be generalised further still and dealt with from two new points of view:

I. The inverse function of $y=\omega(t)=\omega(t;\,\alpha,\,\beta)$ admits of a plain representation

$$\text{(A)} \qquad t = G(y) = q \sum_{\substack{m>-\infty,n=1 \\ n\beta^{-m}<y}}^{\infty} \sum^{\infty}{}' \alpha^{-m} = q \sum_{m>-\infty}^{\infty} a_{m,y}\alpha^{-m} \qquad (0 \leqq y < \infty;\, m > -\log y/\log\beta)$$

where β is an integer, $\alpha>\beta\geqq 2$, $q=(\alpha-\beta)(\beta-1)^{-1}$, and where the dash indicates that n/β should not be an integer; $a_{m,y}$ is the number of positive integers n, not divisible by β and such that, given m, $n\beta^{-m}<y$. From (A) the Lebesgue-Hille-Tamarkin example is obtained by taking $\alpha=3$, $\beta=2$, $y\leqq 1$, the Carleman functions by setting $\beta=2$, $\alpha=3$ or $4, 5, \cdots$; $y\leqq 1$, the Gilman example by taking $y\leqq 1$, α as an integer.

II. The function $\omega(t)=\omega(t;\,\alpha,\,\beta)$ is completely defined by a few trivial conditions and by two functional equations one of which is known in the very special case $\beta=2$, $\alpha=3$(5). Again it satisfies three basic inequalities

$$\text{(B)} \qquad \omega(t+\tau) \leqq \omega(t) + \omega(\tau) \qquad (t \geqq 0,\ \tau \geqq 0),$$

$$\text{(C)} \qquad \text{(i) } \omega(t) \leqq t^{\lambda}; \qquad \text{(ii) } \omega(t) \geqq \left(\frac{\beta-1}{\alpha-1}t\right)^{\lambda} \left(\lambda = \frac{\log\beta}{\log\alpha};\ 0 \leqq t < \infty\right).$$

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(1) That is, $\omega'(t)=0$ for almost all t. H. Lebesgue, *Leçons sur l'intégration*, 2d ed., Paris, 1928.

(2) *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala Universitets Årsscrift, 1923, No. 3, pp. 223–226.

(3) Amer. Math. Monthly vol. 36 (1929) pp. 255–264.

(4) Ann. of Math. (2) vol. 33 (1932) pp. 433–442.

(5) G. H. Hardy and W. Rogosinski, *Fourier series*, Cambridge Tracts, no. 38, 1944, p. 27, line 8 from bottom.

This settles finally the problem of the Lipschitz condition(6), as (B) and (C) imply that

(D) $$0 \leqq \omega(t+h) - \omega(t) \leqq h^{\lambda} \qquad (0 \leqq t < \infty; 0 \leqq h < \infty).$$

There is equality for an infinity of values of t, τ, h. Thus (B)–(D) are best possible results. In connection with problem II the Fourier-Stieltjes transform of $\omega(t)$ $(0 \leqq t \leqq 1)$ is computed(7).

In the second section of the present paper analytic functions are approximated by nearly analytic functions. The analytic functions are required to belong to the class Λ; this is the set of functions $f(z)$ which, for $|z| < 1$, are analytic and such that $f'(z)$ is bounded. The nearly analytic functions satisfy a Lipschitz condition in the region on which they are defined and are, therefore, continuous; their differential coefficient exists in the familiar sense at all points except for a set of surface measure zero. Obviously there are considerable differences between them and analytic functions, for instance with regard to their quasi-poles. They are constructed by means of the function $\omega(t)$. Their theory is given in some detail, since possibly they are of interest in themselves.

Given any two elements $g(z)$ and $h(z)$ of Λ, there exist nearly analytic functions $H_n(z)$ $(n=1, 2, 3, \cdots)$ such that, uniformly for $|z| \leqq 1$, $H_n(z) \to g(z)$ as $n \to \infty$, while $H_n'(z) = h'(z)$ for all n and almost all z (§2.7); explicit formulae are proved for the approximating functions.

Finally results are stated on the approximation of continuous functions of a real variable by "basic functions of bounded variation"(8).

1. **The function** $\omega(t)$. *In this section the main properties of the functions* $G(y)$ *and* $\omega(t)$ *are proved.*

1.1. *The inverse function of* $y=\omega(t)$ *and the functional equations.* The function $t=G(y)$, defined in $(0, \infty)$ by (A), is a jump-function, with discontinuities at the rational points $y_{m,n}=n\beta^{-m}$ $(m=0, \pm 1, \pm 2, \cdots;$ $n=1, 2, \cdots;$ n/β not an integer), and with jump $q\alpha^{-m}$ at $y_{m,n}$. The points $\{y_{n,m}\}$ are dense everywhere in $(0, \infty)$. Hence $G(y)$ increases strictly, and

(6) Cf. Hille and Tamarkin, loc. cit. p. 259, and Gilman, loc. cit. Theorem 2. For $\beta=2$, $t \leqq 1$, (D) follows from a known result. F. Hausdorff (Math. Ann. vol. 79 (1919) pp. 157–179), dealing with exterior measures of "fractional dimensions," associates an exterior measure of sets to any function satisfying some conditions of concavity, and so on, for instance to $t^p (0 < p < 1)$. Then he constructs a function $\phi(t)$ which is closely connected with his exterior measure and is actually identical with $\omega(t; \alpha, 2)$ when we take $p = \log 2/\log \alpha$, ξ_n (see p. 169) equal to α^{-n}, and shows (pp. 170–173) that $\omega(t_2) - \omega(t_1) \leqq (t_2 - t_1)^p$ $(0 \leqq t_1 < t_2 \leqq 1)$. My thanks are due to Professor A. Zygmund for drawing my attention to this classical paper.

(7) For the cases $\beta=2$, $\alpha=3, 4, \cdots$ see Carleman, loc. cit. Cf. Hille and Tamarkin, loc. cit. p. 263.

(8) See §2.8 of the present paper. A non-decreasing function of this kind is called "Scarto elementare" by G. Vitali, Rend. Circ. Mat. Palermo vol. 46 (1922) pp. 388–408; §17.

the inverse function $y=\omega(t)$ is nondecreasing, continuous, and singular[9]; it is constant on the intervals of an open set of measure a for $0\leqq t\leqq a$. We have

$$\begin{aligned} G(y_{m,n}-) = G(y_{m,n}) = t_{m,n}; \quad & G(y_{m,n}+) = t_{m,n} + q\alpha^{-m}, \\ \omega(t) = y_{m,n} \quad \text{for} \quad & t_{m,n} \leqq t \leqq t_{m,n} + q\alpha^{-m}. \end{aligned} \tag{1.11}$$

From (A) we deduce immediately that

$$G\left(\frac{y}{\beta}\right) = \frac{1}{\alpha} G(y) \qquad (0 \leqq y < \infty). \tag{1.12}$$

Again we have

$$G(0) = 0; \quad G(1) = q\left\{\frac{\beta - 1}{\alpha} + \frac{\beta(\beta - 1)}{\alpha^2} + \frac{\beta^2(\beta - 1)}{\alpha^3} + \cdots\right\} = 1, \tag{1.13}$$

since $n\beta^{-m}<1$, $m>0$ and since there are $\beta^{m-1}(\beta-1)$ integers n which are not divisible by β and such that $0<n<\beta^m$. Hence, by (1.12), $G(y)\rightarrow\infty$ as $y\rightarrow\infty$.

Now we take $0<y<1$, $y\neq y_{m,n}$; then we have $m>0$,

$$G(y) + G(1 - y) = q \sum_{n\beta^{-m}<y}^{m,n}{}' \alpha^{-m} + q \sum_{y<\nu\beta^{-m}}^{m,\nu}{}' \alpha^{-m}$$

where $\nu\beta^{-m}=1-n\beta^{-m}$, $\nu=\beta^m-n$ is an integer, while ν/β is not, and $\nu\beta^{-m}<1$. Hence

$$G(y) + G(1 - y) = q \sum_{n\beta^{-m}<1}^{m,n}{}' \alpha^{-m} = G(1) = 1 \quad (y \neq y_{m,n}; 0 < y < 1). \tag{1.14}$$

Finally, for $0<y<1-\beta^{-1}$, we have

$$G\left(y + \frac{1}{\beta}\right) - G(y) = q \sum_{y\leqq n\beta^{-m}<y+\beta^{-1}}{}' \alpha^{-m}.$$

Clearly $m>0$ as $y+\beta^{-1}\leqq 1$. The number of the n's, not divisible by β and such that $\beta^m y\leqq n<\beta^m_y+\beta^{m-1}$, is unity for $m=1$, is $\beta^{m-2}(\beta-1)$ for $m\geqq 2$. Hence

$$\begin{aligned} G\left(y + \frac{1}{\beta}\right) - G(y) &= q\left\{\frac{1}{\alpha} + \sum_{m=2}^{\infty} \frac{\beta^{m-2}(\beta - 1)}{\alpha^m}\right\} \\ &= \frac{\alpha - 1}{\alpha(\beta - 1)} = \frac{q + 1}{\alpha} \qquad \left(0 < y < 1 - \frac{1}{\beta}\right). \end{aligned} \tag{1.15}$$

From (1.12)–(1.15) we deduce the following properties of the inverse function $y=\omega(t)=\omega(t;\,\alpha,\,\beta)$:

[9] H. Kober, *On singular functions of bounded variation*, J. London Math. Soc. vol. 23 (1948) pp. 222–229.

$$(1.16)\quad \begin{aligned}&(a)\quad \omega(0) = 0, \qquad (b)\quad \omega(1) = 1,\\ &(c)\quad \omega(t/\alpha) = \omega(t)/\beta \qquad\qquad (0 \leqq t < \infty),\\ &(d)\quad \omega\left(t + \frac{q+1}{\alpha}\right) = \omega(t) + \frac{1}{\beta}\\ &\qquad\qquad \left(0 \leqq t \leqq 1 - \frac{q+1}{\alpha} = 1 - \frac{\alpha - 1}{\alpha(\beta - 1)}\right),\\ &(e)\quad \omega(t) + \omega(1-t) = 1 \qquad\qquad (0 \leqq t \leqq 1),\end{aligned}$$

observing that $\omega(t)$ is continuous and that $y+1/\beta<1$ implies that $t+(q+1)/\alpha<1$ and vice versa. From (d), or from (c) and (e), respectively, we deduce that

$$(1.17)\quad \begin{aligned}&(d')\quad \omega\left(t + j\frac{q+1}{\alpha}\right) = \omega(t) + \frac{j}{\beta}\\ &\qquad\qquad \left(j = 1, 2, \cdots \beta - 1; 0 \leqq t \leqq 1 - j\frac{q+1}{\alpha}\right),\\ &(e')\quad \omega(t) + \omega(\alpha^j - t) = \beta^j \qquad (j = 0, \pm 1, \pm 2, \cdots; 0 \leqq t \leqq \alpha^j).\end{aligned}$$

We notice that, for $\beta=2$, the equations (c) and (e) together imply (d), as

$$\omega\left(t + 1 - \frac{1}{\alpha}\right) = 1 - \omega\left(\frac{1}{\alpha} - t\right) = 1 - \left(\frac{1}{2} - \omega(t)\right) = \omega(t) + \frac{1}{2}$$

by (e′), with $j=0$ and $j=-1$.

The equation (c) is known for $\alpha=3$, $\beta=2$, $0\leqq t\leqq 1$, (e) is known for the case when α is an integer[10].

1.2. *The Fourier-Stieltjes transform of* $y=\omega(t)$ $(0\leqq t\leqq 1)$ *and the functional equations.* To show that, in substance, the function $y=\omega(t)$ is completely defined by functional equations, we prove the following theorem.

THEOREM 1. *If a function* $y=\psi(t;\alpha,\beta)$ $(\alpha>\beta\geqq 2;$ *β an integer*) (i) *does not decrease in some interval* $0\leqq t\leqq\delta$ *and* (ii) *satisfies* (1.16), (a)–(d), *then it is identical with the function* $y=\omega(t;\alpha,\beta)$.

The proof is based on following theorem.

THEOREM 1′. *Under the conditions of Theorem* 1, *the Fourier-Stieltjes transform*

$$(1.21)\qquad F(x) = \int_0^1 e^{ixt}d\psi(t)$$

is

[10] Gilman, loc. cit.

$$F(x) = e^{ix/2}\prod_{n=0}^{\infty}\frac{\sin(\beta x\alpha^{-n}\Delta/2)}{\beta\sin(x\alpha^{-n}\Delta/2)} \qquad \left(\Delta = \frac{1-\alpha^{-1}}{\beta-1}\right). \tag{1.22}$$

REMARK. When $\beta=2$ then $F(x)$ takes the form

$$F(x) = e^{ix/2}\prod_{n=1}^{\infty}\cos\frac{(\alpha-1)x}{2\alpha^n}\cdot \tag{1.22'}$$

This was proved by Carleman for the case when α is an integer([11]).

Proof. In virtue of (1.16c), (i) implies that $\psi(t)$ does not decrease in $(0, \infty)$. We now consider the $\beta-1$ closed intervals

$$I_j = \langle 1/\alpha + (j-1)\Delta, j\Delta\rangle = \langle a_j, b_j\rangle$$
$$\left(\Delta = \frac{\alpha-1}{\alpha(\beta-1)} = \frac{q+1}{\alpha}; j = 1, 2, \cdots, \beta-1\right). \tag{1.23}$$

On each of these intervals $\psi(t)$ is constant. For $\psi(\alpha^{-1})=\beta^{-1}$ by (b) and (c) while $\psi(\Delta)=\beta^{-1}$ by (a) and (d). Hence $\psi(t)=\beta^{-1}$ on I_1, and $\psi(t)=2\beta^{-1}$ on I_2, $\psi(t)=j\beta^{-1}$ on I_j in consequence of (d′). Hence

$$\int_{a_j}^{b_j} e^{ix(t-1/2)}d\psi(t) = 0 \qquad (j = 1, 2, \cdots, \beta-1). \tag{1.24}$$

Now we have

$$U_j = \int_{b_j}^{a_j+1} e^{ix(t-1/2)}d\psi(t) = \int_0^{1/\alpha} e^{ix(t+j\Delta-1/2)}d\psi(t) \qquad (j = 0, 1, 2, \cdots, \beta-1), \tag{1.25}$$

using (d′) and taking $b_0=0$, $a_\beta=1$. Writing

$$f(x) = \int_0^1 e^{ix(t-1/2)}d\psi(t) \tag{1.26}$$

and using (c), we deduce that

$$U_j = \frac{1}{\beta}\int_0^1 e^{ix(t/\alpha+j\Delta-1/2)}d\psi(t) = \frac{1}{\beta}\exp\left\{ix\left(j\Delta - \frac{\alpha-1}{2\alpha}\right)\right\}f\left(\frac{x}{\alpha}\right). \tag{1.27}$$

Hence

$$f(x) = \sum_{j=0}^{\beta-1}U_j = \beta^{-1}f\left(\frac{x}{\alpha}\right)\exp\left\{\frac{-ix(\alpha-1)}{2\alpha}\right\}\sum_{j=0}^{\beta-1}\exp(ixj\Delta),$$
$$f(x) = u(x)f\left(\frac{x}{\alpha}\right); \qquad u(x) = \frac{\sin(x\beta\Delta/2)}{\beta\sin(x\Delta/2)}\cdot \tag{1.28}$$

[11] Cf. footnote 7.

Thus

$$f\left(\frac{x}{\alpha}\right)=u\left(\frac{x}{\alpha}\right)f\left(\frac{x}{\alpha^2}\right);\quad f\left(\frac{x}{\alpha^n}\right)=u\left(\frac{x}{\alpha^n}\right)f\left(\frac{x}{\alpha^{n+1}}\right)$$

$$(n=0,1,2,\cdots),\tag{1.29}$$

$$f(x)=\prod_{n=1}^{\infty}u\left(\frac{x}{\alpha^n}\right)=\prod_{n=0}^{\infty}\frac{\sin(\Delta\beta\alpha^{-n}x/2)}{\beta\sin(\Delta\alpha^{-n}x/2)}\quad\left(\Delta=\frac{\alpha-1}{\alpha(\beta-1)}\right),$$

as $\alpha>2$ and, by (1.26), $f(x\alpha^{-n})\rightarrow\psi(1)-\psi(0)=1$ $(n\rightarrow\infty)$ for any x. This proves the theorem.

REMARK. It follows from (1.26) that $f(z)$ is an entire function and that, for $z=x+iy$, $|f(z)|\leqq\exp(|y|/2)$.

Now the conditions of the theorem are satisfied by the function $y=\omega(t)$. Hence (1.22) holds for the Fourier-Stieltjes transform of this function. By a known uniqueness theorem and by the continuity of $y=\omega(t)$, the functions $\omega(t)$ and $\psi(t)$ are identical in $\langle 0,1\rangle$ and, by (1.16c), in $\langle 0,\infty\rangle$, which completes the proof of the main theorem.

1.3. *The intervals of invariance of* $\omega(t)$. Before dealing with the inequalities (B) and (C) we must investigate into the position of the intervals on which $y=\omega(t)$ is constant. Let $0\leqq t\leqq 1$, let $y_{m,n}=n\beta^{-m}$ $(m\geqq 1,\ n<\beta^m,\ n/\beta$ not an integer), $t_{m,n}=G(y_{m,n})$, $t'_{m,n}=t_{m,n}+q\alpha^{-m}=G(y_{m,n}+)$ (see 1.11). Then $\omega(t)$ is constant on the interval $I_{m,n}=\langle t_{m,n},\ t'_{m,n}\rangle$ which is said to be of order m $(m=1,2,\cdots)$. The intervals of order one are stated in (1.23). Denoting them now by $I_{1,j}$ $(j=1,2,\cdots,\beta-1)$, we have

$$y=y_{1,j}=\omega\left\{\frac{j(1+q)}{\alpha}\right\}=\frac{j}{\beta}\text{ on }I_{1,j}\quad(j=1,2,\cdots,\beta-1).\tag{1.31}$$

The length of each of the $I_{m,j}$ is $q\alpha^{-m}$, their number is $\beta^m-\beta^{m-1}$ (m fixed). Thus

$$\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}|I_{m,j}|=\sum_{m=1}^{\infty}q\alpha^{-m}(\beta^m-\beta^{m-1})=1.$$

All the intervals of order $k\leqq M-1$ $(M=2,3,4,\cdots)$ can be found this way: set $\eta_N=N/\beta^{M-1}$ where N runs through *all* the numbers $1,2,\cdots,\beta^{M-1}-1$. The number of these intervals is $\beta^{M-1}-1$; their end points are denoted by

$$T_{M-1,N}=G(\eta_N),\qquad T'_{M-1,N}=G(\eta_N+).\tag{1.32}$$

Between any two neighbouring intervals of order $k\leqq M-1$, also between 0 and $T_{M-1,1}$, and between $T'_{M-1,N-1}$ and $T_{M-1,N}=1$ for $N=\beta^{M-1}$, there are $\beta-1$ intervals of order M, with end points

$$\tau_j = G(\eta_N - j\beta^{-M}), \qquad \tau_j' = G(\eta_N - j\beta^{-M}+) = \tau_j + q\alpha^{-M}$$
$$(M,\ N \text{ fixed};\ N = 1, 2, \cdots, \beta^{M-1};\ j = 1, 2, \cdots, \beta - 1).$$

For $\eta_N - j\beta^{-M} = (N\beta - j)\beta^{-M}$, and $N\beta - j$ is not divisible by β. The length of each of these intervals is $q\alpha^{-M}$ by (1.32); writing η for η_N and using (A) we deduce that

$$T_{M-1,N} - \tau_1' = q \sum_{\mu,\nu;\,\eta-\beta^{-M}<\nu\beta^{-\mu}<\eta}' \alpha^{-\mu} = q\left\{\frac{\beta-1}{\alpha^{M+1}} + \frac{\beta(\beta-1)}{\alpha^{M+2}} + \cdots\right\} = \frac{1}{\alpha^M}\cdot$$

For $\eta\beta^{M-1}$ is an integer, while there are no integers between $\eta\beta^{\mu} - \beta^{\mu-M}$ and $\eta\beta^{\mu}$ for $\mu \leqq M$, but $\beta^{l-1}(\beta-1)$ integers for $\mu = M+l$ $(l = 1, 2, \cdots)$. Similarly

$$\tau_j - \tau_{j+1}' = q \sum_{\mu,\nu;\,\eta-(j+1)\beta^{-M}<\nu\beta^{-\mu}<\eta-j\beta^{-M}}' \alpha^{-\mu} = \frac{1}{\alpha^M} \qquad (j = 1, 2, \cdots, \beta - 1),$$

setting $\tau_{j+1}' = T_{M-1,N-1}'$ for $j = \beta - 1$, $N > 1$; $\tau_{j+1}' = 0$ for $j = \beta - 1$ and $N = 1$ (that is, $\eta_N = \eta_1 = \beta^{-M+1}$). Thus we arrive at the following lemma.

LEMMA 1. *In $\langle 0, 1\rangle$ there are $\beta - 1$ intervals of order m $(m > 1)$, each of length $q\alpha^{-m}$, which lie between any two intervals of order smaller than m or between the points $t = 0$ and $t = \alpha^{-m+1}$ or between $t = 1 - \alpha^{-m+1}$ and $t = 1$. The distance between any two neighbouring intervals of order not greater than m $(m \geqq 1)$ is α^{-m}, the left or right end point of the outermost interval on the left or right is $t = \alpha^{-m}$ or $t = 1 - \alpha^{-m}$, respectively.*

1.4. *The inequality* (B). We shall now show that for $0 \leqq t < \infty$, $0 \leqq \tau < \infty$, the inequality (B) holds.

First we deal with a special case and prove that

$$\omega\left(t + \frac{1+q}{\alpha}\right) \leqq \omega(t) + \omega\left(\frac{1+q}{\alpha}\right) \qquad (0 \leqq t < \infty). \tag{1.41}$$

There is equality for $0 \leqq t \leqq 1 - (1+q)\alpha^{-1}$. This follows from (1.16d) as $\omega\{(1+q)\alpha^{-1}\} = \beta^{-1}$.

For $0 < y < \infty$, we have

$$G\left(y + \frac{1}{\beta}\right) - G(y) = q \sum_{y \leqq n\beta^{-m} < y+\beta^{-1}}' \alpha^{-m} \geqq \sum_{y \leqq n\beta^{-m} < y+\beta^{-1};\, m>0}' \alpha^{-m}.$$

By the argument of the proof of (1.15), the latter series converges to $(1+q)\alpha^{-1}$. Hence

$$G\left(y + \frac{1}{\beta}\right) - G(y) \geqq \frac{1+q}{\alpha} = G\left(\frac{1}{\beta}+\right) \qquad (0 < y < \infty). \tag{1.42}$$

As $\omega = G^{-1}$ is nondecreasing (1.42) implies (1.41). There is equality in (1.42) if, and only if, there is no positive integer k such that $y \leqq k < y+\beta^{-1}$, that is, that $\omega(t) \leqq k < \omega(t)+\beta^{-1}$.

By (1.16c), (1.41) implies that

$$\omega\{t+(1+q)\alpha^{-m}\} \leqq \omega(t)+\omega\{(1+q)\alpha^{-m}\} \qquad (0 \leqq t < \infty). \tag{1.41'}$$

Now we show that

$$\omega(t+\zeta') \leqq \omega(t)+\omega(\zeta') \qquad (0 \leqq t < \infty; 0 < \zeta' < 1) \tag{1.43}$$

if ζ' is the right end point of an interval $I_{m,j}$ (see 1.3). By (1.41), we have

$$\omega\left\{t+j\frac{(1+q)}{\alpha}\right\} \leqq \omega\left\{t+\frac{(j-1)(1+q)}{\alpha}\right\}+\omega\left(\frac{1+q}{\alpha}\right)$$
$$(j=1, 2, \cdots, \beta-1),$$

and repeating this procedure (cf. 1.17d') and using (1.31), we deduce (1.43) for the right end points ζ' of the intervals of order one. Suppose now that it holds for the intervals of order smaller than m ($m \geqq 2$). Let $\langle \xi, \xi' \rangle$ ($\xi' < 1$) be the interval of order m which is nearest the right end point of such an interval $\langle \zeta, \zeta' \rangle$. Then we have, using Lemma 1,

$$\xi = \zeta'+\alpha^{-m}; \qquad \xi' = \xi+q\alpha^{-m} = \zeta'+(1+q)\alpha^{-m},$$

and

$$\omega(t+\xi') \leqq \omega(t+\zeta')+\omega\{(1+q)\alpha^{-m}\} \leqq \omega(t)+\omega(\zeta')+\beta^{-m}, \tag{1.44}$$

using (1.43) twice. But $\omega(\zeta')$ is of the form $N\beta^{1-m}=(N\beta)\beta^{-m}$, $\omega(\xi')$ of the form $N'\beta^{-m}$. Hence, applying results of §1.3 to the intervals of order not greater than $m-1$, we have $N'=N\beta+1$

$$\omega(\xi') = (N\beta+1)\beta^{-m} = \omega(\zeta')+\beta^{-m}. \tag{1.45}$$

Combining (1.44) and (1.45) we arrive at (1.43). In a similar way we deal with the right end points of the other $\beta-2$ intervals of order m which lie between $\langle \xi, \xi' \rangle$ and the nearest interval of order not greater than $m-1$ on the right. Between $\tau=0$ and $\tau=\alpha^{-m+1}$ there are $\beta-1$ intervals of order m,

$$I_{m,j} = \langle (j(1+q)-q)/\alpha^m, \; j(1+q)/\alpha^m \rangle \qquad (j=1, 2, \cdots, \beta-1), \tag{1.46}$$

and (1.43) holds for their right end points $j(1+q)\alpha^{-m}$ by (1.41'). Hence (B) is true for the right end point τ of any interval $I_{m,j}$ in (0, 1). Since $\omega(t+\tau)$ is a nondecreasing function of τ, (B) holds for all the points τ of such an interval and, by continuity, for any τ ($0 \leqq \tau \leqq 1$). Using (1.16c), we complete the proof.

REMARK. *By the lemma*[12]: *A function* $y=f(t)$ *of bounded variation* $V_{0,a}f$

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[12] H. Kober, loc. cit. Cf. T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, III.3.30; there $f(t)$ is supposed to be continuous.

over $\langle 0, a\rangle$ *is singular if and only if* $L_{0,a}f$, *the length of the arc joining the points* $\{0; f(0)\}$ *and* $\{a; f(a)\}$, *is equal to* $a+V_{0,a}f$, *the inequality* (B) *is equivalent to* $L_{0,t+\tau}\omega \leqq L_{0,t}\omega + L_{0,\tau}\omega$.

1.5. *Two inequalities*. To prove that $\omega(t) \leqq t^\lambda$ we need two lemmas:

LEMMA 2. *When* $1<\beta<\alpha$, $0\leqq s\leqq 1$, *then*

$$\{1 + (\alpha - 1)s\}^{\log \beta} \geqq \{1 + (\beta - 1)s\}^{\log \alpha}.$$

LEMMA 3. *When* $1\leqq j\leqq \beta$, $0\leqq s\leqq 1$, $0<\log \beta/\log \alpha = \lambda \leqq 1$, *then*

$$(\alpha - js)^\lambda - j(1 - s)^\lambda - (\beta - j) \geqq 0.$$

To prove Lemma 2 we show that the function

$$\phi(s) = \log \beta \log \{1 + (\alpha - 1)s\} - \log \alpha \log \{1 + (\beta - 1)s\}$$

is not negative for $0\leqq s\leqq 1$. We have $\phi(0)=\phi(1)=0$, and it is easily seen that $\phi'(s)$ has not more than one zero. But $\phi'(0)>0$ since

$$\psi(\alpha, \beta) = (\alpha - 1) \log \beta - (\beta - 1) \log \alpha = \phi'(0); \qquad \psi(\beta, \beta) = 0.$$

$$\alpha \frac{\partial\psi(\alpha, \beta)}{\partial\alpha} = \alpha \log \beta - \beta + 1 > \beta \log \beta - \beta + 1 = \int_1^\beta \log u du > 0.$$

Hence, $\phi(s)$ reaches a positive maximum in $(0, 1)$ which proves the lemma.

To prove Lemma 3 we denote the function on the left by $g(s)$. We have

$$(1.51) \qquad (j\lambda)^{-1}g'(s) = (1 - s)^{\lambda-1} - (\alpha - js)^{\lambda-1}$$

and $g(0)=\alpha^\lambda-\beta=0$. The term on the right in (1.51) is not negative; for $\lambda-1\leqq 0$, and $\alpha-js\geqq 1-s$ as $\alpha-1\geqq s(\beta-1)\geqq s(j-1)$. Thus we have proved the lemma.

1.6. *The inequalities* (C). To show that $\omega(t;\alpha,\beta)\leqq t^\lambda$ ($\lambda=\log\beta/\log\alpha$; $0\leqq t<\infty$) we need only prove this for the left end points of the intervals $I_{m,j}$ in $(0, 1)$ (see 1.3). When the intervals are of order one we have $\omega\{(j+jq-q)/\alpha\}=j/\beta$ $(j=1, 2, \cdots, \beta-1)$. Thus we must show that $j(\beta\leqq\{(j+jq-q)/\alpha\}^\lambda$ or, as $\beta=\alpha^\lambda$, that

$$(1.61) \qquad j \leqq \left\{\frac{j(\alpha - 1) - \alpha + \beta}{\beta - 1}\right\}^\lambda.$$

By the substitution $j=1+(\beta-1)s$ $[0\leqq s\leqq(\beta-2)(\beta-1)^{-1}]$, (1.61) reduces to the statement of Lemma 2.

Suppose now that in $(0, 1)$ the inequality is true for the left end points of all the intervals of order smaller than M. Consider the $\beta-1$ intervals of order M lying between neighbouring intervals

$$(1.62) \qquad \langle T_0, T_0'\rangle, \quad \langle T, T'\rangle \qquad (T_0 \geqq \alpha^{1-M} > 0; T_0' = T - \alpha^{-M+1}; T' < 1)$$

of order smaller than M. By Lemma 1, they are

(1.63) $\langle T - j(1+q)/\alpha^M,\ T - (j+(j-1)q)/\alpha^M\rangle \quad (j = 1, 2, \cdots, \beta - 1).$

If $\omega(T) = N\beta^{1-M} = (N\beta)\beta^{-M}$, then we have

$$\omega\left(T - j\frac{1+q}{\alpha^M}\right) = \frac{N\beta - j}{\beta^M}; \qquad \omega(T_0) = \omega(T_0') = \frac{N-1}{\beta^{M-1}}$$
$$(j = 1, 2, \cdots, \beta - 1).$$

We shall show that

$$\{N\beta - j)\beta^{-M} \leqq \{T - j(1+q)\alpha^{-M}\}^\lambda. \tag{1.64}$$

Since $\omega(t) \leqq t^\lambda$ on the intervals of order smaller than M, we have

$$\text{(1.65)} \quad \begin{aligned} &\text{(i)} \quad N\beta^{1-M} = \omega(T) \leqq T^\lambda; \\ &\text{(ii)} \quad (N-1)\beta^{1-M} = \omega(T_0) \leqq \{T - (1+q)\alpha^{1-M}\}^\lambda, \end{aligned}$$

observing that $T_0 \leqq T_0' - q\alpha^{1-M} = T - (1+q)\alpha^{-M}$. Multiplying (1.65) (i) and (ii) by $\beta - j$ and j, respectively, and adding, we have

$$\frac{N\beta - j}{\beta^{M-1}} \leqq (\beta - j)T^\lambda + j\left(T - \frac{1+q}{\alpha^{M-1}}\right)^\lambda \qquad (j = 1, 2, \cdots, \beta - 1). \tag{1.66}$$

Hence we need show that

$$\beta\{T - j(1+q)\alpha^{-M}\}^\lambda - (\beta - j)T^\lambda - j\{T - (1+q)\alpha^{1-M}\} \geqq 0.$$

But this inequality follows from Lemma 3 taking $s = (1+q)\alpha^{1-M}T^{-1}$. The proof holds when T is replaced by unity. Again for $T = \alpha^{1-M}$ there is no interval of order smaller than M on the left; the intervals of order M between $t = 0$ and T are stated above (see 1.46), and the result for their left end points follows from that on the intervals of order one by means of (1.16c). Hence $\omega(t) \leqq t^\lambda$ on all the intervals of order M for any M ($M = 1, 2, \cdots$). By continuity, the inequality holds for $0 \leqq t \leqq 1$, and by (1.16c) for $0 \leqq t < \infty$; which completes the proof of (Ci); that of (Cii) is similar.

REMARK. From the inequalities (B)–(D) we can deduce results on the jump-function $G(y)$, defined by (A); observing that $G(y) = G(y-)$:

$$G(y + z) \geqq G(y) + G(z) \qquad (0 \leqq y < \infty, 0 \leqq z < \infty).$$

$$\frac{\alpha - 1}{\beta - 1} y^{1/\lambda} \geqq G(y) \geqq y^{1/\lambda} \qquad \left(\lambda = \frac{\log \beta}{\log \alpha}; 0 \leqq y < \infty\right).$$

$$G(y + h) - G(y) \geqq h^{1/\lambda} \qquad (0 \leqq y < \infty, 0 \leqq h < \infty; \lambda = \log \beta/\log \alpha).$$

2. **The nearly analytic functions.** *In this section the approximation of elements of* Λ *by nearly analytic functions will be treated.* We start with

2.1. *An inequality.*

LEMMA 4.

$$|\omega(t;\alpha,\beta)-t|\leqq 2/\beta-2/\alpha \qquad \textit{for } 0\leqq t\leqq 1.$$

This need be proved only for the right end points of the intervals on which $\omega(t)$ is constant, or for the left end points. For if $\langle T, T'\rangle$ is such an interval, then, by (1.16e), $\langle 1-T', 1-T\rangle$ is an interval of the same kind. And if the inequality is true for the left end points T and $1-T'$, then it holds for the right end points as, by (1.16e)

$$\omega(T')-T' = 1-\omega(1-T')-T' = (1-T')-\omega(1-T').$$

Since $\omega(t)$ is constant on $\langle T, T'\rangle$, the inequality holds for $T\leqq t\leqq T'$. As the sum of the lengths of all these intervals is unity and as $\omega(t)$ is continuous, the inequality is true for $0\leqq t\leqq 1$.

The right end points of the intervals of order one are (see 1.23; $T_j' = b_j$)

$$T_j' = G\left(\frac{j}{\beta}+\right) = G\left(\frac{j}{\beta}\right)+\frac{1}{\alpha} = \frac{j(\alpha-1)}{\alpha(\beta-1)} \qquad (j=1,2,\cdots,\beta-1).$$

Hence

$$(2.11)\qquad |\omega(T_j')-T_j'| = \left|\frac{j}{\beta}-\frac{j(\alpha-1)}{\alpha(\beta-1)}\right| = \frac{j}{\beta-1}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\leqq\frac{1}{\beta}-\frac{1}{\alpha}.$$

Let T or T', respectively, be the left or right end point of an interval of order not exceeding $m-1$ $(m>1)$, $\omega(T)=N/\beta^{m-1}$ (cf. §1.3; $\omega(t)=\eta_N$) where $N=1$, or $=2, 3,\cdots,\beta^{m-1}-1$, or let $T=1$, with $N=\beta^m$, and suppose that, uniformly with respect to N,

$$(2.12)\qquad |\omega(T)-T|\leqq M_{m-1}.$$

On the left of T, between T and the nearest interval of order not greater than $m-1$, or between T and $t=0$ when $N=1$, there are $\beta-1$ intervals of order m, with left end points

$$t_k = G(N\beta^{1-m}-k\beta^{-m}) = T-k\frac{\alpha-1}{\alpha^m(\beta-1)} \qquad (k=1,2,\cdots,\beta-1).$$

Hence

$$\begin{aligned}|\omega(t_k)-t_k| &= \left|\frac{N\beta-k}{\beta^m}-\left\{T-k\frac{\alpha-1}{\alpha^m(\beta-1)}\right\}\right|\\ &\leqq\left|T-\frac{N}{\beta^{m-1}}\right|+k\left|\frac{1}{\beta^m}-\frac{\alpha-1}{\alpha^m(\beta-1)}\right|\\ &\leqq M_{m-1}+\frac{\beta-1}{\beta^m}-\frac{\alpha-1}{\alpha^m},\end{aligned}$$

if we use (2.12) and observe that $(\beta-1)\beta^{-m}-(\alpha-1)\alpha^{-m}>0$ for $m\geqq 2$, $\alpha>\beta\geqq 2$. Denoting the term on the right by M_m, we have $|\omega(t)-t|\leqq M_m$ for any point t of an interval of order not exceeding m, and

$$M_m \leqq \frac{1}{\beta}-\frac{1}{\alpha}+\frac{\beta-1}{\beta^2}-\frac{\alpha-1}{\alpha^2}+\frac{\beta-1}{\beta^3}-\frac{\alpha-1}{\alpha^3}+\cdots=\frac{2}{\beta}-\frac{2}{\alpha}$$

for any $m\geqq 1$, which completes the proof.

LEMMA 4′.

$$|G(y;\alpha,\beta)-y| \leqq \frac{2}{\beta}-\frac{2}{\alpha} \qquad (0\leqq y\leqq 1).$$

For if $G(y)=t$, then $y=\omega(t)$, and so $|G(y)-y|=|t-\omega(t)|\leqq 2/\beta-2/\alpha$. This is the result required.

2.2. *Definition of the nearly analytic function* $\gamma(z;\alpha,\beta)$. Set

$$\Omega(t)=\omega(t)\quad(0\leqq t\leqq 1);\qquad \Omega(t+1)-\Omega(t)=1 \qquad (-\infty<t<\infty). \tag{2.21}$$

$$\gamma(z;\alpha,\beta)=\gamma(z)=\omega(r)\exp\left\{2i\pi\Omega\left(\frac{\theta}{2\pi}\right)\right\} \qquad (0\leqq r<\infty,\ -\infty<\theta<\infty) \tag{2.22}$$

where $z=re^{i\theta}$. Plainly $\Omega(t)$ is continuous in $(-\infty,\infty)$ as $\omega(0)=0$, $\omega(1)=1$, and $\gamma(z)$ is defined and one-valued for $0\leqq|z|<\infty$ since $\gamma(re^{i\theta})=\gamma(re^{i(\theta+2\pi)})$. From (1.16c) and (C) we deduce that $|\gamma(z)|=\omega(|z|)$,

$$\gamma(\alpha z)=\beta\gamma(z).\qquad \frac{\gamma(z^\alpha)}{|\gamma(z^\alpha)|}=\left\{\frac{\gamma(z)}{|\gamma(z)|}\right\}^\beta \qquad \text{for } 0\leqq\theta\leqq\frac{2\pi}{\alpha}. \tag{2.23}$$

$$|\gamma(z+\zeta)|\leqq|\gamma(z)|+|\gamma(\zeta)| \qquad (0\leqq|z|<\infty,\ 0\leqq|\zeta|<\infty), \tag{2.24}$$

observing that $|\gamma(z+\zeta)|=\omega(|z+\zeta|)\leqq\omega(|z|)+\omega(|\zeta|)=|\gamma(z)|+|\gamma(\zeta)|$ by (B). It is easily shown that there is equality at an infinity of points z, ζ. The function $\gamma(z)$ can be generalized by taking $\omega(t)=\omega(t;\alpha_1,\beta_1)$, $\Omega(t)=\Omega(t;\alpha_2,\beta_2)$ where α_1 and α_2 or β_1 and β_2, respectively, need not be equal.

2.3. *Statements of results on* $\gamma(z;\alpha,\beta)$.

LEMMA 5. *The function* $\gamma(z)$ *has the following properties*:

$$|\gamma(z;\alpha,\beta)-z|\leqq(2+4\pi)\left(\frac{1}{\beta}-\frac{1}{\alpha}\right). \qquad (0\leqq z\leqq 1). \tag{I}$$

(II) *Its variation over any radius of the unit-circle is not greater than unity, over any circle* $|z|=R$ $(0<R\leqq 1)$ *not greater than* 2π.

(III) *It satisfies the Lipschitz condition*

$$|\gamma(z_1) - \gamma(z_2)| \leqq C_0 |z_1 - z_2|^\lambda \qquad (|z_1| \leqq 1, |z_2| \leqq 1)$$

where $C_0 = 1 + \pi 2^{2-3\lambda}$, $\lambda = \log \beta / \log \alpha$. *It is, therefore, continuous.*

(IV) *There is an infinity of sectors* $S_{j,l}$ $(j=1, 2, \cdots; l=1, 2, \cdots)$ *of rings in* $|z| < 1$, *of total area* $\sum |S_{j,l}| = \pi$, *such that* $\gamma(z)$ *is constant on any* $S_{j,l}$.

(IV′) *The derivative* $\gamma'(z)$ *exists and vanishes for almost all* z $(|z| \leqq 1)$.

(V) *When* $\alpha < \beta^2$, *then, given* $\delta > 0$, *there is a non-enumerable set of circles* $|z| = R$ $(0 < R < \delta)$, *any of them having a non-enumerable set of quasi-poles* ζ *on its circumference, that is, points* ζ *such that* $\{\gamma(z) - \gamma(\zeta)\}(z-\zeta)^{-1} \to \infty$ *as* $z \to \zeta$, *no matter along which path* z *approaches* ζ.

2.4. **Proof of** (I), (II), **and** (III). To deduce (I), we take $0 \leqq \theta < 2\pi$. We have

$$|\gamma(z) - z| \leqq \omega(r) |e^{2i\pi\omega(\theta/2\pi)} - e^{2i\pi\theta/2\pi}| + |\omega(r) - r|$$
$$\leqq 2\pi \left|\omega\left(\frac{\theta}{2\pi}\right) - \frac{\theta}{2\pi}\right| + |\omega(r) - r| \leqq (4\pi + 2)\left(\frac{1}{\beta} - \frac{1}{\alpha}\right),$$

using Lemma 4 and the inequality

$$(2.41) \qquad |e^{it} - e^{i\tau}| \leqq |t - \tau| \qquad (t, \tau \text{ real}).$$

Now we prove (II). Fixing θ, we have, with respect to the variable r,

$$V_{0,1}\gamma = V_{0,1}\omega = \omega(1) - \omega(0) = 1.$$

Fixing r and taking $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, we have

$$V_{0,2\pi}\gamma \leqq 2\pi\omega(r)\left\{\text{u.b.} \sum_{j=0}^{n-1} \left|\omega\left(\frac{\theta_{j+1}}{2\pi}\right) - \omega\left(\frac{\theta_j}{2\pi}\right)\right|\right\}$$
$$= 2\pi\omega(r)\{\omega(1) - \omega(0)\} \leqq 2\pi.$$

To prove (III), we take $z_j = r_j e^{i\theta_j}$, $0 \leqq \theta_j < 2\pi$, $j = 1, 2$; $z_3 = r_2 e^{i\theta}$; and $0 < r_2 \leqq r_1$. By (D) we have

$$(2.42) \qquad |\gamma(z_1) - \gamma(z_3)| = \omega(r_1) - \omega(r_2) \leqq (r_1 - r_2)^\lambda \leqq |z_1 - z_2|^\lambda.$$

Again

$$|\gamma(z_3) - \gamma(z_2)| = \omega(r_2) |e^{2i\pi\omega(\theta_1/2\pi)} - e^{2i\pi\omega(\theta_2/2\pi)}|$$
$$(2.43) \qquad \leqq 2\pi\omega(r_2) \left|\omega\left(\frac{\theta_1}{2\pi}\right) - \omega\left(\frac{\theta_2}{2\pi}\right)\right|,$$
$$|\gamma(z_3) - \gamma(z_2)| \leqq (2\pi)^{1-\lambda}\omega(r_2) |\theta_1 - \theta_2|^\lambda.$$

Suppose first that $|\theta_1 - \theta_2| \leqq \pi$; then we have $\sin |(\theta_1 - \theta_2)/2| \geqq |(\theta_1 - \theta_2)|/\pi$,

$$(2.44) \qquad |\theta_1 - \theta_2| \leqq (\pi/2r_2) |z_3 - z_2| \qquad (|\theta_1 - \theta_2| \leqq \pi),$$

and as, by (Ci), $\omega(r) \leqq r^\lambda$

$$(2.45)\qquad |\gamma(z_3) - \gamma(z_2)| \leqq 2^{1-2\lambda}\pi |z_3 - z_2|^\lambda \leqq 2^{1-2\lambda}\pi |z_1 - z_2|^\lambda,$$

observing that $r_2 \leqq r_1$, $|z_2| = |z_3|$, $\arg z_3 = \arg z_1$ and, therefore,

$$(2.46)\qquad |z_3 - z_2| \leqq |z_1 - z_2|.$$

Suppose now that $\pi < \theta_2 - \theta_1 < 2\pi$, and take $\chi_1 = \theta_1 + 2\pi$; then

$$0 < \chi_1 - \theta_2 < \pi, \quad 1 \leqq \frac{\chi_1}{2\pi} < 2, \; z_3 = r_2 e^{i\chi_1}, \quad \Omega\left(\frac{\chi_1}{2\pi}\right) = \omega\left(\frac{\theta_1}{2\pi}\right) + 1,$$

$$\begin{aligned}\Omega\left(\frac{\chi_1}{2\pi}\right) - \Omega\left(\frac{\theta_2}{2\pi}\right) &= \left\{1 - \omega\left(\frac{\theta_2}{2\pi}\right)\right\} + \omega\left(\frac{\theta_1}{2\pi}\right) = \omega\left(1 - \frac{\theta_2}{2\pi}\right) + \omega\left(\frac{\theta_1}{2\pi}\right)\\ &\leqq \left(1 - \frac{\theta_2}{2\pi}\right)^\lambda + \left(\frac{\theta_1}{2\pi}\right)^\lambda = \left(1 - \frac{\theta_2}{2\pi}\right)^\lambda + \left(\frac{\chi_1}{2\pi} - 1\right)^\lambda\\ &\leqq 2^{1-2\lambda}\pi^{-\lambda}(\chi_1 - \theta_2)^\lambda,\end{aligned}$$

if we use (1.16e), (D) and the inequality $u^\lambda + v^\lambda \leqq 2^{1-\lambda}(u+v)^\lambda$ $(0 \leqq \lambda \leqq 1, u > 0, v > 0)$. Hence, using (2.44) and (2.46), we deduce that

$$(2.47)\qquad \begin{aligned}|\gamma(z_3) - \gamma(z_2)| &\leqq 2\pi\omega(r_2)\left|\Omega\left(\frac{\chi_1}{2\pi}\right) - \Omega\left(\frac{\theta_2}{2\pi}\right)\right|\\ &\leqq (2\pi)^{1-\lambda} 2^{1-\lambda}\omega(r_2)\left|\frac{\pi}{2r_2}(z_3 - z_2)\right|^\lambda \leqq 2^{2-3\lambda}\pi |z_1 - z_2|^\lambda.\end{aligned}$$

The result holds for $\pi < \theta_1 - \theta_2 < 2\pi$. Combining (2.45) or (2.47), respectively, and (2.42) we deduce (III) for $|z_j| > 0$ $(j = 1, 2)$. The inequality holds for $z_1 = 0$, or $z_2 = 0$, since $|\gamma(z)| = \omega(r) \leqq r^\lambda$; which completes the proof. We remark that, by (2.23), the result holds in the whole plane.

2.5. **Proof of** (IV) **and** (IV′). We consider the intervals $\langle T_j, T_j' \rangle$ $(0 < T_j < T_j' < 1; j = 1, 2, \cdots)$ on which $\omega(t)$ is constant, taken in any order. We have

$$(2.51)\qquad \sum_{j=1}^{\infty} (T_j' - T_j) = 1.$$

The function $\gamma(z)$ is constant on any ring-sector $S_{j,l}$, where

$$T_j \leqq r \leqq T_j', \qquad 2\pi T_l \leqq \theta \leqq 2\pi T_l' \qquad (j = 1, 2, \cdots; l = 1, 2, \cdots).$$

The area of $S_{j,l}$ is $|S_{j,l}| = \pi(T_j' - T_j)(T_l' - T_l)$; hence

$$(2.52)\qquad \sum_{j=1, l=1}^{\infty,\infty} |S_{j,l}| = \pi \sum_{j=1, l=1}^{\infty,\infty} (T_j' - T_j)(T_l' - T_l) = \pi.$$

Now we construct a region $P_{j,l}$ interior to $S_{j,l}$ such that, given $\epsilon>0$,

$$| P_{j,l} | \geqq | S_{j,l} | - \epsilon 2^{-j-l}.$$

Everywhere on $P_{j,l}$ the derivative $\gamma'(z)$ exists and vanishes identically, and we have

$$\sum_{j=1,l=1}^{\infty} | P_{j,l}| \geqq \sum_{j=1,l=1}^{\infty} | S_{j,l}| - \epsilon \to \pi \qquad \text{as } \epsilon \to 0. \tag{2.53}$$

Hence (IV) and (IV′) are true.

2.6. **Proof of** (V). We need

LEMMA 6(13). *If the function $\sigma(t)$ $(a \leqq t \leqq b)$, nondecreasing and not reducing to a constant, is continuous and singular, then $\sigma'(t)=\infty$ at a non-enumerable set of points of (a, b).*

Given $\delta>0$, we have $\omega(\delta)>\omega(0)=0$ in virtue of (Cii). Hence there exists a non-enumerable set E of points r of $(0, \delta)$ such that $\omega'(r)=\infty$. Also there is a similar set $\mathcal{E}$ of points θ of $(0, 2\pi)$ such that $\omega'(\theta/2\pi)=\infty$. We take $r\in E$ $(r>0)$, $\theta \in \mathcal{E}$, $z=re^{i\theta}$, $z_1=r_1 e^{i\theta_1}$ $(r_1>0)$ and form

$$U(z_1) = \frac{\gamma(z_1)-\gamma(z)}{z_1-z} = \frac{s_1(e^{i\psi_1}-e^{i\psi})+e^{i\psi}(s_1-s)}{z_1-z} \tag{2.61}$$

where

$$s=\omega(r), \quad s_1=\omega(r_1); \qquad \psi = 2\pi\omega\left(\frac{\theta}{2\pi}\right), \quad \psi_1 = 2\pi\omega\left(\frac{\theta_1}{2\pi}\right). \tag{2.62}$$

Given any large $M>0$,we can find $\epsilon>0$ such that, for $|z_1-z|<\epsilon$,

$$\frac{s_1-s}{r_1-r} = D > M, \qquad \frac{\psi_1-\psi}{\theta_1-\theta} = d > M. \tag{2.63}$$

We have

$$U(z_1)e^{i(\theta-\psi)} = \frac{ie^{i(\psi_1-\psi)/2}ds_1\mu(\theta_1-\theta)+(r_1-r)D}{ie^{i(\theta_1-\theta)/2}r_1\nu(\theta_1-\theta)+r_1-r} = \frac{X}{Y} \tag{2.64}$$

where

$$\mu = \sin\frac{\psi_1-\psi}{2}\Big/\frac{\psi_1-\psi}{2}, \qquad \nu = \sin\frac{\theta_1-\theta}{2}\Big/\frac{\theta_1-\theta}{2},$$

$$|Y|^2 = (r_1-r)^2 + \nu^2 r r_1(\theta_1-\theta)^2, \tag{2.65}$$

(13) Cf. Hille and Tamarkin, loc. cit. p. 258, footnote 2; and Gilman, loc. cit., where much more detailed results are proved. For the general case see H. Kober, loc. cit. Cf. S. Saks, *Theory of the integral*, 2d. ed., Warsaw, 1937, p. 128, lines 14–12 from bottom.

$$\Re(X\overline{Y}) = D(r_1 - r)^2 + dr_1 s_1 \mu\nu \cos\frac{\psi_1 - \psi - \theta_1 + \theta}{2}(\theta_1 - \theta)^2$$

$$(2.66) \qquad -\frac{D}{2} r_1\nu^2(r_1 - r)(\theta_1 - \theta)^2$$

$$-\frac{d}{2} s_1\mu^2(r_1 - r)(\theta_1 - \theta)(\psi_1 - \psi) = U + V - W - Z.$$

Using (2.65) and (D) we have

$$\left|\frac{W}{Y^2}\right| = \left|\frac{r_1\nu^2(s_1 - s)(\theta_1 - \theta)^2}{2Y^2}\right| \leqq \left|\frac{r_1\nu^2(r_1 - r)^\lambda(\theta_1 - \theta)^2}{2rr_1\nu^2(\theta_1 - \theta)^2}\right|$$

$$= \frac{|r_1 - r|^\lambda}{2r} \to 0 \qquad (z_1 \to z).$$

$$\left|\frac{Z}{Y^2}\right| = \frac{s_1\mu^2|r_1 - r|(\psi_1 - \psi)^2}{2|Y|^2} \leqq \frac{2^{1-2\lambda}\pi^{2-2\lambda}s_1\mu^2|r_1 - r|(\theta_1 - \theta)^{2\lambda}}{(r_1 - r)^2 + \nu^2 rr_1(\theta_1 - \theta)^2}$$

$$\leqq 2^{-2\lambda}\pi^{2-2\lambda}s_1(rr_1)^{-1/2}\mu^2\nu^{-1}(\theta_1 - \theta)^{2\lambda-1} \to 0 \qquad (z_1 \to z),$$

as $2\lambda = \log \beta^2/\log \alpha > 1$. Finally we deduce that

$$\frac{U + V}{|Y|^2} \geqq \frac{M\{(r_1 - r)^2 + r_1 s_1\mu\nu(\theta_1 - \theta)^2 \cos\}}{(r_1 - r)^2 + rr_1\nu^2(\theta_1 - \theta)^2}$$

$$\geqq M \min\left\{1, \frac{s_1\mu \cos}{r\nu}\right\} \to \infty \qquad (z_1 \to z).$$

For $(u^2+av^2)(u^2+bv^2)^{-1} \geqq \min(1, a/b)$ $(a>0,\ b>0,\ u$ and v real$)$, $M\to\infty$ as $z_1\to z$, while $s_1\to\omega(r)>0$, $\mu\to 1$, $\cos(\psi_1-\psi-\theta_1+\theta)/2\to 1$, $\nu\to 1$; therefore $\Re(X/Y)\to\infty$ as $z_1\to z$, $\Re\{U(z_1)e^{i(\theta-\psi)}\}\to\infty$,

$$\left|\frac{\gamma(z_1) - \gamma(z)}{z_1 - z}\right| \to \infty \qquad \text{as } z_1 \to z.$$

Thus (V) is proved. A similar result holds for $\gamma(z)$ in any sector $A \leqq \theta \leqq B$ of the ring $a \leqq r \leqq b$ $(0 < a < b \leqq 1)$ if $\omega(B/(2\pi)) > \omega(A/(2\pi))$ and $\omega(b) > \omega(a)$.

2.7. *The main results on the class* Λ. Suppose that $f(z)$ belongs to Λ. Write

$$C_f = \underset{|z|<1}{\text{u.b.}} |f'(z)|.$$

The function $f(z)$ exists on the circle $|z| = 1$ and is continuous for $|z| \leqq 1$. Let

$$\eta(z; \alpha, \beta) = f\{\gamma(z; \alpha, \beta)\}.$$

When $f(z)$ does not reduce to a constant then $\eta(z)=\eta(z;\alpha,\beta)$ is a nearly analytic function, that is, it has the properties (II)–(V), *with the following alterations* (cf. §2.3):

The variations, referred to in (II), *are not greater than C_f or $2\pi C_f$, respectively.*

The constant $1+\pi 2^{2-3\lambda}$, occurring in (III), *is to be replaced by $C_f(1+\pi 2^{2-3\lambda})$.* This is shown with the aid of the inequality

$$(2.71)\qquad |f(w_2)-f(w_1)| = \left|\int_{w_1}^{w_2} f'(z)dz\right| \leqq C_f|w_2-w_1|.$$

We now prove the following theorem.

THEOREM 2. *When both $g(z)$ and $h(z)$ belong to Λ and $g(z)-h(z)$ does not reduce to a constant then there is a sequence of functions $H_n(z)$ $(n=1, 2, \cdots)$ such that, uniformly for $|z|\leqq 1$, $H_n(z)\rightarrow g(z)$ as $n\rightarrow\infty$, while $H_n'(z)=h(z)$ for all n and for almost all z. The functions $H_n(z)$ possess the properties* (II), (III), *and* (V), *but the constants are C_g+2C_h and $2\pi(C_g+2C_h)$ in* (II), $(C_g+2C_h)\cdot(1+2\pi 2^{2-3\lambda})$ *in* (III). *They have the properties* (IV) *and* (IV′) *if, and only if, $h(z)$ vanishes identically.*

Proof. Set $f(z)=g(z)-h(z)$, let β be a fixed integer $(\beta\geqq 2)$, $\alpha=\beta+n^{-1}$,

$$H_n(z) = f\{\gamma(z;\alpha,\beta)\} + h(z).$$

Using (2.71) and (I), we have

$$|H_n(z)-g(z)| = |f\{\gamma(z)\}-f(z)| \leqq C_f|\gamma(z)-z|$$
$$\leqq C_f(2+4\pi)\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) = \frac{C_f(2+4\pi)}{\beta(n\beta+1)}$$

which tends to zero as $n\rightarrow\infty$, uniformly for $|z|\leqq 1$. On the other hand, we have

$$H_n'(z) = \frac{df\{\gamma(z)\}}{dz} + h'(z) = f'\{\gamma(z)\}\gamma'(z) + h'(z) = h'(z)$$

for almost all z in $|z|<1$ since, by (IV′), $\gamma'(z)=0$ almost everywhere. We notice that the condition $\alpha<\beta^2$ (see V) is satisfied. Thus we have proved the theorem.

The nearly analytic functions appear to be adapted to illustrate the Besicovitch-Saks-Zygmund Theorem (cf. Saks, loc. cit., p. 197).

2.8. *On continuous functions of a real variable.* A continuous function $y=F(t)$, of bounded variation over $\langle 0, a\rangle$ and such that the sum of the lengths of its intervals of invariance is a, is said to be basic; for instance $\omega(t)$. When $F(t)$ $(0\leqq t\leqq a)$ is continuous, then it can be approximated uniformly by basic functions of bounded variation(14). This follows at once from Theorem 2 by means of the Weierstrass approximation theorem; it is analogous to the uniform approximation by step-functions, which are singular functions as well as the basic functions of bounded variation. We can obtain similar results in some other way, and we state them without proof.

When $y=f(x)$ $(0\leqq x\leqq a)$ *is continuous then the sum of the lengths of the intervals on which the continuous function*

$$g_{\alpha,\beta}(x) = f\left\{a\omega\left(\frac{x}{a};\alpha,\beta\right)\right\} \qquad (0\leqq x\leqq a)$$

is constant equals a, *and* $g_{\alpha,\beta}(x)=0$ *for almost all* x. *Yet* $g_{\alpha,\beta}(x)$ *is a basic function of bounded variation if, and only if,* $f(x)$ *is of bounded variation; otherwise,* $g_{\alpha,\beta}(x)$ *is not of bounded variation*(15). *In any case, we have*

$$g_{\alpha,\beta}(x) \to f(x) \qquad \text{as } \alpha\to\beta,$$

uniformly for $0\leqq x\leqq a$. *Similarly, uniformly for* $0\leqq x\leqq a$,

$$f\left\{aG\left(\frac{x}{a};\alpha,\beta\right)\right\} \to f(x) \qquad \text{as } \alpha\to\beta.$$

The latter two results follow from the Lemmas 4 and 4′ without difficulty.

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(14) Also there exists a result corresponding to Theorem 2.

(15) Compare this with Vitali's example, loc. cit., §25.